# Towards Automatic Verification of Affine Hybrid System Stability*

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## 1. Introduction

Stability is a very important property of feedback control systems. For linear feedback control systems, methods for proving or disproving system stability are well known. They basically amount to eigenvalue computations. For more general classes of feedback control systems, however, these methods are not applicable. A very general method is based on Lyapunov theory [7]. It uses a so-called Lyapunov function which can be regarded as generalized "energy function" of the system under investigation. If energy converges to zero over time then the system state will converge towards an equilibrium state.

In the scope of the AVACS project,[1] we are concerned with *automatically* proving hybrid system stability. In a first phase of the project, we focus on a special subclass of hybrid systems, namely *piecewise affine* hybrid systems. Hybrid systems contain both, discrete states and continuous-time dynamics. Piecewise affine hybrid systems require the continuous-time dynamics to be affine for each discrete state. A powerful method of verifying stability for such systems is based on extensions of Lyapunov theory, as proposed by Pettersson [9], Johansson *et. al.* [6] or Branicky [4]. These approaches still require system dependent inputs which could – up to now– only be provided by a human proof designer based on his or her intuition and expert knowledge of the problem domain.

The work presented in this paper reports on our current status with respect to achieve *full automatization* of the stability verification task. The goal is to be able to prove stability of piecewise affine hybrid systems "by the push of a button" and without any further knowledge provided by a human user. In the future, we plan to extend and to complement the approach in order to cope with more general non-linear systems, possibly through approximation.

## 2. Preliminaries

**Definition 1** A *continuous-time, autonomous hybrid system (HS)* is a system of the form

$$\begin{aligned} \dot{x}(t) &= f(x(t), m(t)) \\ m(t^+) &= \phi(x(t), m(t)) \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the *continuous state vector*[2] of the HS at time instant $t$ and $m(t) \in \mathcal{M} := \{1, .., M\}$ is its *discrete state*. $m(t^+)$ denotes the updated discrete state right after time instant $t$. $\mathcal{H} = \mathbb{R}^n \times \mathcal{M}$ is called *hybrid state space*. The function $f : \mathcal{H} \to \mathbb{R}^n$ describes the behavior of the continuous state and $\phi : \mathcal{H} \to \mathcal{M}$ describes the behavior of the discrete state of the HS. $f$ is assumed being continuously differentiable. $\mathcal{S}_{m_1,m_2} = \{x : \phi(x, m_1) = m_2\}, m_1 \neq m_2$, denotes the *switch set* from discrete state $m_1$ to discrete state $m_2$. A solution $x(t)$ of the HS *(1)* for a particular tuple $(x(0), m(0))$ of starting points is called a *trajectory*. A HS is called *piecewise affine* if for each $m \in \mathcal{M}$, $f(x, m)$ is affine in $x$, i.e. there exist a matrix $A_m \in \mathbb{R}^{n \times n}$ and a vector $b_m \in \mathbb{R}^n$, such that $f(x, m) = A_m x + b_m$ for all $x$. $\square$

The most widespread notion for stability of autonomous systems is Lyapunov stability [8].

**Definition 2** *([7])* A system[3] is *globally asymptotically stable* in $\underline{0}$ ($\underline{0}$ being the $n$-dimensional zero vector), iff

- (A1) for all $\epsilon > 0$ there exists a $\delta > 0$, such that for all initial continuous states $x(0)$ with $\|x(0)\| < \delta$, $\|x(t)\| < \epsilon$ holds for all $t \geq 0$.
- (A2) for all initial continuous states $x(0)$: $x(t) \to \underline{0}$, $t \to \infty$

If only (A1) holds then the HS is *globally stable* in $\underline{0}$. A globally asymptotically stable system is called *globally exponentially stable* if for all $\epsilon > 0$ there exist $\delta, k_1, k_2 \in \mathbb{R}$, all greater than $0$, such that $\|x(0)\| < \delta$ implies $\|x(t)\| \leq k_1 e^{k_2 t} \|x_0\|$ for all $x(0)$ and $t \geq 0$. $\square$

For non-hybrid systems (equivalent to HS with a single discrete state), a means to prove global asymptotic stability is via a Lyapunov function. The Lyapunov function converges towards $0$ for $t \to \infty$.

**Theorem 1** *(Lyapunov [8])*
Let $\dot{x} = f(x)$ be a continuous-time (non-hybrid) system with $f(\underline{0}) = \underline{0}$. If there exists a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$, such that

- (L1) $V(x)$ is positive definite, i.e. $V(\underline{0}) = 0$ and $V(x) > 0$ for all $x \neq 0$

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1 Please visit `www.avacs.org` for more details.

2 If the value of $t$ is not important, we will simplify $x(t)$ to $x$ and $m(t)$ to $m$.

3 The system may also be hybrid.

(L2) $\dot{V}(x)$ is negative definite, i.e. $\dot{V}(\underline{0}) = 0$ and $\dot{V}(x) < 0$ for all $x \neq 0$

(L3) $V(x) \to \infty$ for $\|x\| \to \infty$

then the system is globally asymptotically stable in $\underline{0}$. $V$ is then called a *Lyapunov function* of the system. □

This theorem can be generalized for hybrid systems by partitioning the state space into so-called *regions*. For each region, it is required to find a "pseudo-Lyapunov function," that fulfills conditions *(L1)* and *(L2)* for that particular region. If we assure that the "composite pseudo-Lyapunov function" consisting of the composition of the region-specific "pseudo-Lyapunov functions" does not increase when crossing region boundaries, this implies global exponential stability. This is formalized in the following theorem.

**Theorem 2** *(Pettersson/Lennartson [10])*
Let $\mathcal{R} = \{R_1, ..., R_N\}$ be a partitioning of $\mathcal{H}$. For each pair of regions $R_i$ and $R_j$, define the *transition set* $T_{i,j} := \{(x, m) \,|\, \exists t > 0 : (x(t - \epsilon), m(t - \epsilon)) \in R_i$ and $(x(t + \epsilon), m(t + \epsilon)) \in R_j$, when $\epsilon \to 0, \epsilon > 0\}$. Let $I_\Lambda \subseteq \{1, ..., N\} \times \{1, ..., N\}$, with $(i, j) \in I_\Lambda$ iff $T_{i,j} \neq \{\}$. Assume that for each $R_i$ we have a continuously differentiable function $V_i : \mathbb{R}^n \to \mathbb{R}$. If there exist $\alpha, \beta, \gamma \in \mathbb{R}$, all positive, such that:

(H1) for each $R_i$: $\alpha x^T x \leq V_i(x) \leq \beta x^T x$ for all $x$ with $\exists m : (x, m) \in R_i$

(H2) for each $R_i$: $\frac{\partial V_i(x)}{\partial x} f(x, m) \leq -\gamma x^T x$ for all $(x, m) \in R_i$

(H3) for each pair of regions $R_i$ and $R_j$ with $(i, j) \in I_\Lambda$: $V_j(x) \leq V_i(x)$ for all $x \in T_{i,j}$

then the HS is globally exponentially stable in $\underline{0}$. □

## 3. LMI-Based Verification

For piecewise affine (PWA) systems, it is advisable to restrict the search for such a set of "pseudo-Lyapunov functions" $V_i$ to a parameterized subset. In particular, we choose *quadratic expressions*: polynomials of degree 2. This restriction allows us to employ convex optimization methods [3], while still retaining an adequate degree of flexibility.

**Definition 3** A *quadratic expression* is a function of the form $g(x) = x^T P x + 2p^T x + \pi$, $P \in \mathbb{R}^{n \times n}, p \in \mathbb{R}^n, \pi \in \mathbb{R}$. Then, $g(x) = \tilde{x}^T \tilde{P} \tilde{x}$ holds, with

$$\tilde{x} := \begin{bmatrix} x \\ 1 \end{bmatrix}, \quad \tilde{P} := \begin{bmatrix} P & p \\ p^T & \pi \end{bmatrix}$$

□

This allows us to reduce Theorem 2 to a linear matrix inequality problem for any given partitioning. Since linear matrix inequalities (LMIs) are special cases of convex optimization problems, they can be solved efficiently [2, 3].

**Theorem 3** *(Pettersson [9])*

(P1) For each $R_i$, $1 \leq k \leq \kappa_i, \kappa_i > 0$, find sets of $\tilde{Q}_i^k \in \mathbb{R}^{n \times n}$ with $\tilde{x}^T \tilde{Q}_i^k \tilde{x} \geq 0$ for all $x$ with $\exists m : (x, m) \in R_i$.

(P2) For each $R_i$ and each $m$ with $\exists x : (x, m) \in R_i$, $1 \leq k \leq \kappa_{i,m}, \kappa_{i,m} > 0$, find sets of $\tilde{Q}_{i,m}^k \in \mathbb{R}^{n \times n}$ with $\tilde{x}^T \tilde{Q}_{i,m}^k \tilde{x} \geq 0$ for all $x$ with $(x, m) \in R_i$.

(P3) For each pair of regions $R_i$ and $R_j$ with $(i, j) \in I_\Lambda$, $1 \leq k \leq \kappa_{i,j}, \kappa_{i,j} \geq 0$, find sets of $\tilde{Q}_{i,j}^k \in \mathbb{R}^{n \times n}$ with $\tilde{x}^T \tilde{Q}_{i,j}^k \tilde{x} \geq 0$ for all $x \in T_{i,j}$.

Let $I$ be the $n \times n$ identity matrix and define

$$\tilde{I} := \begin{bmatrix} I & \underline{0} \\ \underline{0}^T & 1 \end{bmatrix}, \quad \tilde{A}_m := \begin{bmatrix} A_m & b_m \\ \underline{0}^T & 0 \end{bmatrix}$$

then

$$\begin{aligned} \dot{x}(t) &= A_m x(t) + b_m \\ m(t^+) &= \phi(x(t), m(t)) \end{aligned}$$

is globally exponentially stable in $\underline{0}$ if there exist real numbers $\alpha > 0, \mu_i^k \geq 0, \nu_i^k \geq 0, \vartheta_{i,m}^k \geq 0$ and $\eta_{i,j}^k \geq 0$, such that the following LMI problem has a solution:

Minimize $\beta$ subject to

$$\alpha \tilde{I} + \sum_{k=1}^{\kappa_i} \mu_i^k \tilde{Q}_i^k \leq \tilde{P}_i \leq \beta \tilde{I} - \sum_{k=1}^{\kappa_i} \nu_i^k \tilde{Q}_i^k, \quad 1 \leq i \leq N \tag{2}$$

$$\tilde{A}_m^T \tilde{P}_i + \tilde{P}_i \tilde{A}_m + \sum_{k=1}^{\kappa_{i,m}} \vartheta_{i,m}^k \tilde{Q}_{i,m}^k \leq \tilde{I}, \quad 1 \leq i \leq N, \exists x \in \mathbb{R}^n : (x, m) \in R_i \tag{3}$$

$$\tilde{P}_j + \sum_{k=1}^{\kappa_{i,j}} \eta_{i,j}^k \tilde{Q}_{i,j}^k \leq \tilde{P}_i, \quad (i, j) \in I_\Lambda \tag{4}$$

□

For each $k$, the sets $\{x | x^T \tilde{Q}_i^k x \geq 0\}$ over-approximate $\{x | \exists m : (x, m) \in R_i\}$, the sets $\{x | x^T \tilde{Q}_{i,m}^k x \geq 0\}$ over-approximate $\{x | (x, m) \in R_i\}$ and the sets $\{x | x^T \tilde{Q}_{i,j}^k x \geq 0\}$ over-approximate $T_{i,j}$. The LMIs result from a relaxation called $\mathcal{S}$-procedure [11] that allows local definiteness-constraints, so that the equivalents of *(H1)-(H3)* need only be fulfilled locally. Therefore, each $\tilde{Q}$ matrix corresponds to a quadratic expression.

Informally, Theorem 3 implies that one has to define a suitable partitioning, calculate $I_\Lambda$ and find appropriate $\tilde{Q}$ matrices. Then it is possible to solve the LMIs through convex optimization in order to obtain a "family" of $\tilde{P}_i$ matrices. These matrices represent the quadratic expressions that form the "pseudo-Lyapunov functions" $V_i$ required by Theorem 2. The entire procedure will be explained next.

## 4. Automatization of the Verification Task

There are four steps which have to be performed for an automatization of the verification task. The first step, described in Section 4.1, is partitioning the state space into a number of regions for which "pseudo-Lyapunov functions" must be found. The second step deals with finding the transitions between the regions. This is shown in Section 4.2. Step three describes the conversion of the regions found so far as well as the transitions into an LMI problem. This is presented in Section 4.3. Finally, step four (given in Section 4.4) is concerned with the solution of the LMI problem.

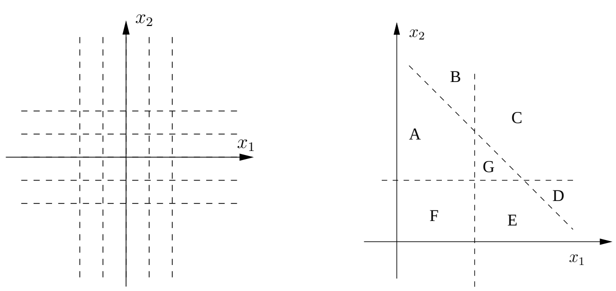
### 4.1. Partitioning of the Hybrid State Space

In our current research, we restrict ourselves to switch sets that are defined by hyperplanes since for those switch sets the transitions between regions can be calculated more efficiently. In our approach, the partitioning of the hybrid state space results in regions that only span a single discrete state, i.e. for each discrete state the continuous subspace of the hybrid state space will be partitioned separately. However, in this paper, we assume that this partitioning will be the same for all discrete states.

We follow two different approaches in order to partition the hybrid state space. The first approach uses a grid for partitioning. This leads to cubic regions and infinite cuboid-like regions. See Figure 1(a) for an example. The second approach uses the hyperplanes of the switch sets to create a partitioning. Figure 1(b) shows a partitioning into regions A to G using three hyperplanes. In both approaches, the boundary between two neighboring regions can be described by an intersection of a hyperplane

$$S = \{x \,|\, c^T x + d = 0\}, \quad c \in \mathbb{R}^n,\, d \in \mathbb{R} \tag{5}$$

and one of the two regions. The regions created by one of the two approaches may be partitioned further into smaller subregions. This may become necessary if global exponential stability cannot be shown using a certain partitioning although one assumes that the respective hybrid system is indeed globally exponentially stable. On the other hand, it may be possible to merge regions in order to increase efficiency of the verification task, e.g. one could use a single region for a certain discrete state spanning the whole continuous state space.




(a) Partitioning using a grid.

(b) Partitioning according to the switch sets.

**Figure 1. Two different approaches of partitioning the state space.**

### 4.2. Identifying the Region Transitions

Let us once again refer to Theorem 2. Generally, calculating $I_\Lambda$ may not be easy. Note that using a superset of $I_\Lambda$ in place of $I_\Lambda$ in formula (4) will also result in a proof of global exponential stability if a solution is found. Therefore, it is safe to over-approximate $I_\Lambda$ by starting with $\{1, ...N\} \times \{1, ...N\}$ and successively removing all region transitions that are known to be impossible. The closer this over-approximation is to $I_\Lambda$, the higher the chance of finding a solution of the LMI.

Luckily, for affine functions $f(x, m)$, it is easy to determine whether transitions between two regions $R_i$ and $R_j$ are possible in either direction or only in a single one. The procedure we are using is as follows: for each discrete state $m$ with affine vector field $f(x, m) = A_m x + b_m$ identify the hyperplane $S$ defined by formula (5) which separates $R_i$ and $R_j$, and identify the hyperplane $T = \{x \,|\, (c^T A_m)x + c^T b_m\}$. $T$ defines the points at which $\dot{x}$ is a vector parallel to $S$. The intersection of $S$ and $T$ gives the set of points at which the direction of the trajectories crossing $S$ changes. Refer to Figure 2 for an example in $\mathbb{R}^2$ in which $S \cap T$ contains only a single element $\bar{x}$. Let $B_{i,j}$ be the boundary between $R_i$ and $R_j$. If the intersection of $T$ and $B_{i,j}$ is empty then the boundary can only be crossed in at most one direction. Whether $T \cap B_{i,j}$ is empty or not, can be decided by solving a linear program. The particular manner in

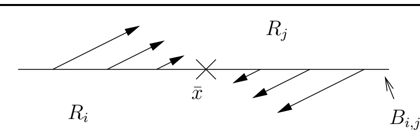



**Figure 2. Trajectories cross $B_{i,j}$ in either direction.**

which the transitions between regions having different discrete states are calculated, depends on the approach used to partition the continuous state space. If the "hyperplane approach" is used, it only needs to be checked whether the discrete state changes on a boundary between two regions. If the "grid approach" is pursued, it must be checked for each region whether there are switching hyperplanes intersecting with it. This, again, can be tested using linear programs.

### 4.3. Formulating Regions and Transitions as LMI

In this section, it is described how the $\tilde{Q}$ matrices of Theorem 3 can be identified. For simplicity, we assume that the partitioning presented in Section 4.1 is the same for every discrete state. Thus, we omit the index $m$ in the $\tilde{Q}^k_{i,m}$.

Either of our approaches to partition the hybrid state space uses regions $R_i$ that can be represented as an intersection of half-planes $\{x \mid c^T x + d \geq 0\}$. In [9] it is proposed to create quadratic expressions for every half-plane and for each combination of two of them. This results in $\kappa_i = ((\sigma_i + 1)\sigma_i)/2$ expressions where $\sigma_i$ is the number of half-planes for region $R_i$. The matrices $\tilde{Q}^k_i, k \in \{1, \ldots, \kappa_i\}$ are then calculated in an analogous manner to the calculation of matrix $\tilde{P}$ in Definition 3. Cubic regions can be handled in a simpler way: one determines the smallest ellipsoid containing the cubic region which can be described by a simple quadratic expression. This results in fewer quadratic expressions while reducing flexibility.

For the boundaries $B_{i,j}$, as needed in formula (4), another approach to create quadratic expressions is used. Assume that $S$ as in formula (5) is the hyperplane belonging to the boundary $B_{i,j}$. Pettersson [9] suggests a number of $\kappa_{i,j} = n + 1$ quadratic expressions to represent a hyperplane. As a region boundary often is a bounded subset of such a hyperplane, one can think of weakening the particular constraint by adding one or more other quadratic expressions weighted by new variables $\eta^k_{i,j}$. For example, for the boundary between two neighboring cubic regions, one may extend the set of quadratic expressions by the single quadratic expression describing the smallest ellipsoid enclosing one of the regions.

### 4.4. Solving the LMI Problem

After creating the quadratic expressions for regions and boundaries, the resulting LMI is presented to a solver. So far, we have used the two semidefinite programming (SDP) solvers, namely SDPA [5] and CSDP [1]. According to Theorem 3, our PWA system is globally exponentially stable if the LMI has a solution. Otherwise, we cannot draw a simple conclusion. If no solution is found by the solver then there are three possibilities: (1) the PWA system is unstable, (2) the PWA system is stable but the partitioning is not appropriate or not enough transitions have been excluded, and (3) the solver did not find a solution of the LMI although there exists one.

## 5. Conclusion and Ongoing Work

We have shown how heuristic methods and convex optimization can be combined in order to show global exponential stability of piecewise affine hybrid systems. With suitable heuristics this approach leads to a fully automatic polynomial-time algorithm. The approach has plenty of potential for further improvements and generalizations. In principle, it can deal with arbitrarily shaped switch sets or even nondeterministic switching. Furthermore, it is possible to show stability of a large class of non-linear systems by "embracing" them with two equally partitioned PWA systems: the two PWA systems must behave in such a way that the behavior of the system under investigation will always stay between these two PWA systems. Then, if identical "pseudo-Lyapunov" functions for both PWA systems can be identified, the "embraced" system is guaranteed to exhibit the same stability properties. Discrete-time systems can also be treated this way, with a slight modification of the LMIs and the algorithm for calculating the region transitions. A third direction for future work is local stability, i.e. a stability property that holds only if the initial state lies within a certain set of states. Then, if it is known that the complement of the desired "stable region" cannot be reached by the HS, it is possible to restrict the partitioning accordingly. Consequently, LMIs are more likely to have solutions.

Before such extensions can be made, a deep understanding of the partitioning process is required. As of now, we are analyzing different partitioning approaches and the effects of different choices of quadratic expressions for over-approximating the regions.

A back-end that translates the partitioning and system information into a standard format for convex programming has already been implemented. We are confident being able to report on full automatization of PWA system stability not too far in the future.

## References


[1] B. Borchers. CSDP, a C library for semidefinite programming. *Optimization Methods and Software*, 10:613–623, 1999.

[2] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*. SIAM, 1994.

[3] S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.

[4] M. S. Branicky. Multiple Lyapunov functions and other analysis tools for switched and hybrid systems. *IEEE Transactions on Automatic Control*, 43(4), April 1998.

[5] K. Fujisawa, M. Kojima, K. Nakata, and M. Yamashita. SDPA (SemiDefinite Programming Algorithm). `http://grid.r.dendai.ac.jp/sdpa/`.

[6] M. Johansson and A. Rantzer. Computation of piecewise quadratic Lyapunov functions for hybrid systems. *IEEE Transactions on Automatic Control*, 43, 1998.

[7] R. E. Kalman and J. E. Bertram. Control system analysis and design via the "second method" of Lyapunov. *Transactions of the ASME, Journal of Basic Engineering*, pages 371–400, 1960.

[8] M. A. Lyapunov. Problème général de la stabilité du movement. *Ann. Fac. Sci. Toulouse*, 9:203–474, 1907. (Translation of a paper published in Comm. Soc. math. Kharkow, 1893, reprinted in Ann. math. Studies No. 17, Princeton University Press, 1949).

[9] S. Pettersson. *Analysis and Design of Hybrid Systems*. PhD thesis, Chalmers University of Technology, Gothenburg, 1999.

[10] S. Pettersson and B. Lennartson. Controller design of hybrid systems. In *LNCS*, volume 1201, pages 240–254, 1997.

[11] V. Yakubovich. S-procedure in nonlinear control theory. *Vestnik Leningrad Univ.*, 4:73–93, 1977.